Measurement and meaningfulness in Decision Modeling

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Chapter 2

Outline

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Numbers

- Today, mathematics are used in all the fields of human activity, not only as a tool to make calculations, but also in the education, the methodology and everyone's way of thinking: we all reason in terms of measures, percentages, ratios, logical deductions, statistics,...
- The constitution of a system of numbers is already a mathematical theory, with many rules, conventions or axioms. These rules can be different depending on what these numbers represent.

Numbers

- A first use of numbers is of course numbering (first, second, . . .), i.e., giving a list in a certain order (ordinal aspect of numbers).
- A second use of natural numbers(positive integers) is to count objects (cardinal aspect); in this perspective, some basic operations can be introduced, such as addition and subtraction.
- However, the main use of numbers resides in one of the most natural activities of humans: measuring.

Numbers and measurement

- Measuring allows to quantify phenomena, to make calculations in order to understand, to foresee and to manage our environment.
- Measuring weights, lengths or volumes is necessary in commercial transactions.
- Measuring heat, duration or flow is useful to describe physical phenomena.
- Measuring wealth, unemployment or production allows to analyze economy.
- Measuring pollution, noise or vegetation density is necessary in environmental management

Numbers and measurement

- Numbers are used to measure many other things: speed, age, density, score, social impact, economic index, probability, possibility, credibility, preference intensity, latitude, date, earthquake intensity, popularity, prediction, . . .
- Manipulating "numbers" in social sciences, as most of the decision aiding tools try to do, raises the question of measuring human or social characteristics, such as satisfaction, risk aversion, preference, group cohesion, etc.
- It seems clear that the numbers representing measures cannot always be treated in the same way because the underlying information can be completely different from one context to another.

Introduction to measurement theory

- Measurement: assignment of numbers to attributes of the natural world; It is central to all scientific inference
- Measurement theory concerns the relationship between measurements and reality
- Its goal is ensuring that inferences about measurements reflect the underlying reality we intend to represent.

Introduction to measurement theory

- The key principle of measurement theory is that theoretical context, the rationale for collecting measurements, is essential to defining appropriate measurements and interpreting their values
- Theoretical context determines the scale type of measurements and which transformations of those measurements can be made without compromising their meaningfulness.
- Despite this central role, measurement theory is almost unknown by some practitioners of Decision Modeling, and its principles are frequently violated.

Outline

Example (The race)

The arrival order in a race is the following: Alfred, Bill, Carl, David, Ernest, Franz, Gilles, Henry, Isidore and John.

- Team *a*: Alfred, David, Franz and John
- Team *b*: Bill, Carl, Ernest, Gilles, Henry, Isidore
- The duration of the race has been registered, in seconds, yielding for each runner, giving the numbers in this Table:

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

- The purpose is to compare these two teams and, if possible, to decide which team is the best.

Example (The race (times in seconds))

Team a=$\{A; D; F; J\}$ and Team b=$\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

On the basis of these numbers, are the following assertions valid (true or false)?

1. The mean time of Team b is higher than the mean time of Team a;
2. The second best (lowest) time in Team b is lower than the second best time in Team a;
3. The mean time computed on the basis of all the runner' results, is beaten by three runners of Team a and three runners of Team b;
4. The median [a] time, calculated on the basis of all of the runners' results, is exceeded by two runners of Team a and three runners of Team b;
5. The third best time in Team a is lower than the times of three runners of Team b;

[a] The median of a set of numbers is a value x such that there as many numbers greater than x than number smaller than x

Example (The race (times in seconds))

Team $a=\{A;D;F;J\}$ and Team $b=\{B;C;E;G;H;I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

On the basis of these numbers, are the following assertions valid?

6. The worst time in Team b is more than 1.2 times the best time in Team a;
7. The difference between the worst time in Team a and the worst time in Team b is 12.5 times the difference between the best time in Team a and the best time in Team b;
8. The sum of the two best times in Team a is higher than the sum of the two best times in Team b;
9. The difference between the two best times in Team a is triple the difference between the two best times in Team b;
10. If we consider the three best times, Team b is more often represented than Team a;

Example (The race (times in seconds))

Team a=$\{A; D; F; J\}$ and Team b=$\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

On the basis of these numbers, are the following assertions valid?

11. The sum of the three best times in Team a is higher than the sum of the two best times in Team b;
12. The mean time of Team b is 1.015 times the mean time of Team a;
13. The ratio between the worst and the best times is higher in Team a than in Team b;
14. In Team a, the square of the worst time is 1.6 times the square of the best time;
15. The difference between the best and the worst times in Team a is equal to 11.5.

Example (The race (times in seconds))

Team $a=\{A; D; F; J\}$ and Team $b=\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
43.5	43.7	44.2	45	47	48	52	52.1	52.5	55

- Convert all the times into minutes
- After this changement, the previous fifteen assertions remain true or false?

Outline

Example (The weather (Celsius degrees))

Temperatures were measured at noon in two European countries, during respectively 10 and 8 consecutive days. The results, in Celsius degrees, are :

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

- On the basis of these observations, how could we help a tourist choose a country for his holidays?

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

On the basis of these numbers, are the following assertions valid (true or false)?

1. The mean temperature in country *a* is higher than the mean temperature in country *b*;
2. The second highest temperature in country *a* is higher than the highest temperature in country *b*;
3. The mean temperature calculated on the basis of all the measures in both countries, is exceeded seven times in country *a* and three times in country *b*;
4. The median value, calculated on the basis of all the measures in both countries, is exceeded four times in country *a* and once in country *b*;
5. The fourth highest temperature in country *a* is higher than the temperatures in country *b* during 5 days;

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

On the basis of these numbers, are the following assertions valid?

6. The highest temperature in country *a* is more than 1.5 times the lowest temperature in country *b*;
7. The difference between the highest temperature in country *a* and the highest temperature in country *b* is four times the difference between the lowest temperature in country *a* and the lowest temperature in country *b*;
8. The sum of the two highest temperatures in country *a* is larger than the sum of the two highest temperatures in country *b*;
9. The difference between the two highest temperatures in country *a* is two times the difference between the two highest temperatures in country *b*;
10. If we consider the five highest temperatures in table 3.2, country *a* is more often represented than country *b*;

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

On the basis of these numbers, are the following assertions valid?

1. The sum of the three highest temperatures in country *a* is larger than the sum of the four lowest temperatures in country *b*;
2. The mean temperature in country *a* is 1.1 times the mean temperature in country *b*;
3. The ratio between the highest and the lowest temperatures is larger in country *a* than in country *b*;
4. In country *a*, the square of the highest temperature is 2.37 times the square of the lowest temperature;
5. The difference between the highest and the smallest temperatures in country *a* is equal to 7.

Example (The weather (Celsius degrees))

	1	2	3	4	5	6	7	8	9	10
a	20	16	15	14	14	15	13	15	16	18
b	14	12	13	15	14	13	15	16	-	-

- Convert into Fahrenheit degrees the number (in order to limit the number of decimals, we simply multiply by 1.8 and added 32)
- After this changement, the previous fifteen assertions remain true or false?

Outline

Example (The race again)

Suppose now that the only available information is the ranking of the runners and that numbers have been associated to them in decreasing order of the arrivals:

Team $a = \{A; D; F; J\}$ and Team $b = \{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
10	9	8	7	6	5	4	3	2	1

On the basis of these numbers, are the following assertions valid (true or false)?

1. The mean of team a is greater than the mean of Team b;
2. The second highest number in Team b is bigger than the second highest number in Team a;
3. Two runners of Team a and three runners of Team b have a number that is bigger than the mean of the whole set of runners;
4. Two runners of Team a and three runners of Team b have a number which is bigger than the median of the whole set of runners;
5. The third highest number in Team a is greater than the numbers of three runners of Team b;

Example (The race again)

Team a=$\{A; D; F; J\}$ and Team b=$\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
10	9	8	7	6	5	4	3	2	1

On the basis of these numbers, are the following assertions valid (true or false)?

6. The greatest number in Team a is less than two times the number of the third runner in Team b;
7. The difference between the numbers of the best runners of Teams a and b is equal to the difference between the numbers of the last runners of these teams;
8. The sum of the two highest numbers in Team a is equal to the sum of the two highest numbers in Team b;
9. The difference between the numbers of the first and the second runners of Team a is triple the difference between the numbers of the first and the second runners of Team b;
10. Team b has more runners among the three highest numbers than Team a, and also among the five highest;

Example (The race again)

Team a={A; D; F; J} and Team b={B; C; E; G; H; I}

A	B	C	D	E	F	G	H	I	J
10	9	8	7	6	5	4	3	2	1

On the basis of these numbers, are the following assertions valid (true or false)?

11. The highest number in Team a is larger than the sum of the three lowest numbers in Team b but smaller than the sum of the four lowest numbers in Team b;
12. The mean of Team a is 1.17 times the mean of Team b;
13. The ratio between the second and the third highest numbers is larger in Team a than in Team b;
14. In Team b, the square of the highest number is 20.25 times the square of the smallest one;
15. The difference between the greatest and the smallest number in Team a is equal to 9.

Example (The race again)

In fact, in this example, the only relevant information is the ranking of the runners, and there is no reason to privilege one numerical representation over another.

Consider, for instance, this numerical representation

Team $a=\{A; D; F; J\}$ and Team $b=\{B; C; E; G; H; I\}$

A	B	C	D	E	F	G	H	I	J
100	90	80	10	9	8	7	6	5	0

On the basis of these new numbers, the previous fifteen assertions remain true or false?

Outline

Example (The expert's advice)

Suppose that an expert evaluated social projects in a city by assigning numbers to them in function of what he considers as their chance of success and their global interest for the city. The scale is [0, 20] and the higher the evaluation, the higher the quality of the project.

A	B	C	D	E	F	G	H	I	J
17	16	14	12	10	10	9	5	3	2

Example (The expert's advice)

A	B	C	D	E	F	G	H	I	J
17	16	14	12	10	10	9	5	3	2

On the basis of these numbers, are the following assertions valid (true or false)?

1. project A is the best;
2. project E is two times better than project H;
3. the difference between projects A and B is less than that between D and E;
4. the differences between B and C and between C and D are equal;
5. four projects are "below the mean" (which is equal to 10);
6. if two projects can be chosen, the pair {B,C} is better than {A,D} (as the sum of their evaluations is higher).

Example (The expert's advice)

A	B	C	D	E	F	G	H	I	J
17	16	14	12	10	10	9	5	3	2

- In this example, the numbers are associated to subjective evaluations (by the expert) and not to some "objective facts" such as times, temperatures or ranking, as was the case in the previous examples.
- This means that the reliability of a conclusion based on these numbers depends on the type of information they really support.
- This can be the subject of additional assumptions or can be obtained by a dialog with the expert on how he has built his evaluations.
- Such a dialog could reveal, for example, that his evaluations of "bad" projects were only very roughly made (so that the difference between H and I has no meaning at all), or that he really hesitated to consider that A is better than B, while he was sure that C is much better than D.

Example (The expert's advice)

A	B	C	D	E	F	G	H	I	J
17	16	14	12	10	10	9	5	3	2

- Moreover, if this expertise has to be merged with other information, the decision maker may want to take into account the inevitable imprecisions of such subjective evaluations by considering that a difference of one point between two projects can be ignored.
- In this case, the following Table of evaluations could be considered as equivalent to to the previous one for the purpose of comparing projects

A	B	C	D	E	F	G	H	I	J
17	17	15	12	10	10	10	0	0	0

On the basis of these new numbers, the previous six assertions remain true or false?

Outline

Definitions

- "Evaluating" an object consists in associating an element of a numerical scale (a subset of real numbers) to it, according to some conventions as, for example, the choice of a measurement instrument
- The evaluation of an object along a numerical scale is supposed to characterize or to represent a particular information about certain aspects of this object (weight, temperature, age, number of votes, development of a country, air quality, hospitals rankings, hotels ranking, etc.)
- Changing the conventions leads to changing the evaluations of the objects.

Definitions

- Different numerical scales are considered as being "equivalent" if they support (represent) the same information about the considered objects: we will call them "info-equivalent".
- "Admissible transformations" means "transformations into info-equivalent numerical scales".

Types of scale: the ordinal scale

- A scale is *ordinal* if its admissible transformations are all strictly increasing transformations;

Types of scale: the interval scale

- It is *an interval scale* if its admissible transformations are all positive affine transformations of the form

$$\phi(x) = \alpha\ x + \beta \text{ (with } \alpha > 0)$$

In this case, the scale is univocally determined by the choice of an origin and a unit

Types of scale: the ratio scale

- It is a ratio scale if its admissible transformations are the positive homothetic transformations of the form

$$\phi(x) = \alpha\ x \text{ (with } \alpha > 0)$$

In this case, the scale is univocally determined by the choice of a unit, the origin being "naturally fixed"

Types of scale: the absolute scale

- The absolute scale does not accept any admissible transformation (except the identity)
- Ex: a counting or a probability scale

Remark

In many cases, it is not possible to characterize the transformations between info-equivalent numerical scales in an analytical way

see for example "the expert's advice"

Meaningful assertion

- In classical measurement theory, an assertion is declared to be meaningful if its truth value is unchanged when admissible transformations are applied to the scales used in the assertion
- More generally (when the admissible transformations are not identifiable), we will say that an assertion is meaningful if its truth value is unchanged when the numerical scales used in the assertion are replaced by info-equivalent scales

Types of scale: the ordinal scale

- A scale is *ordinal* if its admissible transformations are all strictly increasing transformations;
- Let be $\succsim$ a binary relation on X such that

$$\forall a, b \in X, \begin{cases} a \succ b \Leftrightarrow f(a) > f(b) & a \text{ “is preferred to” } b \\ a \sim b \Leftrightarrow f(a) = f(b) & a \text{ “is indifferent to” } b \end{cases}$$

  - $\succ$ and $\sim$ are invariant for any strictly increasing transformation of the scale of X (leading to an info-equivalent scale).
  - The representation of $\succ$ and $\sim$ leads to an ordinal scale.
  - Every assertion based on these relations can thus be considered as “meaningful”.

Types of scale: the interval scale

- It is *an interval scale* if its admissible transformations are all positive affine transformations of the form $\phi(x) = \alpha\ x + \beta$ (with $\alpha > 0$)
- Let be $\succsim$ a binary relation on X such that

$$\forall a, b, c, d \in X, \begin{cases} a \succ b \Leftrightarrow f(a) > f(b) & a \text{ "is preferred to" } b \\ a \sim b \Leftrightarrow f(a) = f(b) & a \text{ "is indifferent to" } b \\ (a, b) \succ^* (c, d) \Leftrightarrow f(a) - f(b) > f(c) - f(d) \\ (a, b) \sim^* (c, d) \Leftrightarrow f(a) - f(b) = f(c) - f(d) \end{cases}$$

  - $(a, b) \succ^* (c, d)$ means "the preference of a over b is stronger than that of c over d" and $(a, b) \sim^* (c, d)$ means "the preference of a over b is as strong as that of c over d".
  - The representation of $\succ$, $\succ^*$, $\sim$ and $\sim^*$ leads to an interval scale.
  - $\succ$, $\succ^*$, $\sim$ and $\sim^*$ are invariant for any positive affine transformation of X (leading to an info-equivalent scale), so that assertions solely based on them are meaningful.

Types of scale: the ratio scale

- It is a ratio scale if its admissible transformations are the positive homothetic transformations of the form

$$\phi(x) = \alpha\ x \text{ (with } \alpha > 0)$$

  In this case, the scale is univocally determined by the choice of a unit, the origin being "naturally fixed"

Be careful

- As we see, depending on the scale type (i.e. depending on the information supported by the scale), some caution is necessary in the manipulation and the interpretation of the numbers if we want to obtain meaningful conclusions based on these numbers.
- A conclusion that is true using a given scale but that is meaningless (not meaningful) for this type of scale is completely dependent of the particular scale which is considered, has no character of generality and is thus, probably, of very limited interest.
- **It can even be dangerous because of the tendency of humans to generalize ideas without sufficient precautions.**
- The analysis of scale types allows to detect manipulations (mathematical operations) which can lead to meaningless conclusions

Be careful

- Scale types are not "naturally given" in decision aiding, even for physical measures, and that every use of numbers must be accompanied by some precisions on the information they are supposed to support
- Meaningfulness theory is an important tool for the analysts in order to avoid the development of completely arbitrary decision aiding procedures
- In evaluation and decision problems, the nature of the numbers used is partially in the hands of the analyst: it mainly depends on the purpose of the decision aiding process and on the future steps of the process (is it really useful to build a ratio scale if the next step only exploits the ordinal properties of the numbers?).
- The role of the analyst is to be sure that all the operations are compatible with his choice, from the assessment of the numbers to their interpretation, including the mathematical manipulations of these numbers.

Be careful

- Even if the numbers are useful, their presence in a "model" does not guarantee that it is a formal model. In a sense, the ease of use of the numbers may be a pitfall since it can lead to instrumental bias.
- Another confusion is often made between the term "qualitative" and the absence of numerical information.
  - The color of an object is typically qualitative but can be represented by a number (the wave length).
  - On the contrary, the expression "a small number of students" does not contain any number but is certainly not qualitative. It represents a quantity.

Reference

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